Monty McGovern lecture on "Slicing and dicing the flag variety" on 07/18/10 at 2PM

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1 Introduction

Let G be a complex algebraic reductive group. Let B be a Borel subgroup of G. Let K be the fixed points under G of an involution θ. We want to study K-orbits on G/B, and their closures. There are finitely many such orbits. They may be parameterized combinatorially. This has been done in all the classical cases, but in the exceptional cases, there are no "nice" lists available. The big questions are :

For each orbit closure $\overline{O}$, is $\overline{O}$ smooth as an algebraic variety? Is it rationally smooth? (a variety is rationally smooth if the 0th and top local cohomology of the variety don't vanish and the top cohomology is one dimensional) Or neither?

First, some general results (following Brion) on more general things :

Let X be an algebraic variety, and H a complex algebraic group acting on X. Assume T is a nontrivial torus contained in H that fixes a point $x \in X$.

To decide whether X is (rationally) smooth at x, we can replace X by something smaller. This is the "slicing" in the title of the talk. Assume that $S \subset X$ is a subvariety that satisfies :

i) x is isolated in $S \cap Hx$.

ii) T is a maximal torus in $H_x =$ stabilizer of x. (Note that T acts on S.

iii) The morphism $H \times S \to X$, sending (h, y) to hy, is smooth at $(1, x)$.

iv) x is *attractive* for the T-action on S, which means that all weights of T on the tangent space T_xS lie on one open side of a hyperplane. (think of positive roots in a root system)

Theorem 1.1. *(Brion) Under these hypotheses, X (rationally) smooth at x if and only if S is (rationally) smooth at x.*

So we can therefore consider a smaller S, and study (rational) smoothness at x by considering x in S as opposed to x in X. Sometimes, S is still too big, and so we perform the same procedure above on S.

Theorem 1.2. *(Brion) S is (rationally) smooth at x if and only if*

i) S admits a rationally smooth punctured neighborhood of x

ii) For each subtorus T' of T of codimension 1, $S^{T'}$ (the fixed points) is rationally smooth at x.

iii) $\sum_{T' \subset T} dim_X S^{T'} = dim_X(S)$, where T' range over the subtori of codimension 1. $S^{T'}$ denotes fixed points of T' on S.

Specialize down to the case $K = O(n)$ and $G = GL(n)$ (This is the last open classical case. In all other cases, Monty knows the answer). Orbits X are parameterized by involutions $\pi \in S_n$. More precisely, a flag $V_0 \subset \cdots \subset V_n$ is in O_π if and only if the rank of $(\cdot,\cdot)_{V_i \times V_j}$ equals the cardinality of $\{k \leq i : \pi(k) \leq j\}$ for all i, j, where $(\cdot,\cdot)$ is the standard inner product on $\mathbb{C}^n$, and where $dim(V_j) = j$. We have that $\overline{O_\pi} \subset \overline{O_{\pi'}}$ if and only if $\pi \subset \pi'$ in the reverse Bruhat order.

Example: For $GL(2)$, then there are two involutions : They are the identity permutation and the nonidentity permutation in S_2. The identity permutation corresponds to the open orbit, and the nonidentity permutation corresponds to the closed orbit.

For simplicity, let's assume that $n = 2m$ is even. Fix a basis $e_1, ..., e_{2m}$ of $\mathbb{C}^{2m}$ such that $(e_i, e_j) = 1$ if $i+j = 2n+1$, and $(e_i, e_j) = 0$ otherwise. We will now define our slice S. Fix $x = eB$, the identity coset. It's a fact that to check rational smoothness, we just need to show rational smoothness at x. To define S, restrict to flags $V_0 \subset \cdots \subset V_n$ such that $V_i = span(b_1, ..., b_n)$ where $b_1 = a_{11}e_1 + \cdots + a_{1m}e_m + e_{2m}$, $b_2 = a_{12}e_1 + \cdots + a_{2m}e_m + e_{2m-1}$, ..., $b_m = a_{1m}e_1 + \cdots + a_{mm}e_m + e_{m+1}$, $b_{m+1} = a_{1,m+1}e_1 + \cdots + a_{m-1,m+1}e_{m-1} + e_m, \cdots b_{2m} = e_1$.

Then the matrix $a_{i,j}$ where $1 \leq i, j \leq m$, is a generic symmetric matrix, and the rest of the $a_{i,j}$ give a generic unipotent anti-triangular matrix. The definition of S is the set of all such flags just described, where $a_{i,j}$ run over all possible values.

Examples: The 2 smallest interesting cases in $GL(4)$. Look at the orbits given by the permutations (2 1 4 3), which is rationally singular, and (1 3 2 4), which is singular but rationally smooth. In the orbit (2 1 4 3), you get $a_{11} = 0$ and that the determinant of the matrix

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{12} & a_{22} & 1 \\ a_{13} & 1 & 0 \end{pmatrix}$$

has determinant zero, so $a_{13}(2a_{12} - a_{13}a_{22}) = 0$. Thus, we get rationally singular at eB. The other orbit is given by (1 3 2 4). Here, the condition to belong to $S \cap \overline{O}$, where $\overline{O}$ is the orbit corresponding to (1 3 2 4), is given by the vanishing of the determinant of $\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$. This orbit is therefore rationally smooth but not smooth.

Conjecture 1.3. *(McGovern)*
Let n be even. Given π, form its Bruhat graph BG_π, consisting of all involutions $\mu \leq \pi$. Then there is going to be an edge from μ to ν if and only if $\nu = t\mu t \neq \mu$, where t is a transposition of two indices OR $\nu = t\mu$, where t is a transposition of two indices such that $t\mu t = \mu$. Then, $\overline{O_\pi}$ is rationally smooth at O_c implies that $deg(O_c) = dim(\overline{O_\pi}) - dim(\overline{O_c})$, where O_c means the closed orbit.. Then the conjecture is that in practice, for n even, this is a necessary and sufficient condition for rational smoothness.

For n odd, there are problems.

This degree condition applies not merely to O_c, but to any vertex conjugate to O_c (below O_π) under the cross action.

Recently, McGovern has learned that for $GL(7)$, this condition (in the last sentence) holds, yet the orbit closure is rationally singular. For (1 3 2 5 4), the condition holds for O_c, but fails for another O_μ, so the orbit closure is rationally singular. For (2 1 3 7 6 5 4), the condition holds for all O_μ, so the orbit is rationally singular.